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The Secrets of Mental Math

Course Guidebook

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Professor Arthur T. Benjamin is a Professor of Mathematics at Harvey Mudd College. He graduated from Carnegie Mellon University in 1983, where he earned a B.S. in Applied Mathematics with university honors. He received his Ph.D. in Mathematical Sciences in 1989 from Johns Hopkins University, where he was supported by a National Science Foundation graduate fellowship and a Rufus P. Isaacs fellowship. Since 1989, Professor Benjamin has been a faculty member of the Mathematics Department at Harvey Mudd College, where he has served as department chair. He has spent sabbatical visits at Caltech, Brandeis University, and the University of New South Wales in Sydney, Australia.

In 1999, Professor Benjamin received the Southern California Section of the Mathematical Association of America (MAA) Award for Distinguished College or University Teaching of Mathematics, and in 2000, he received the MAA Deborah and Franklin Tepper Haimo National Award for Distinguished College or University Teaching of Mathematics. He was also named the 2006–2008 George Pólya Lecturer by the MAA.

Professor Benjamin's research interests include combinatorics, game theory, and number theory, with a special fondness for Fibonacci numbers. Many of these ideas appear in his book (coauthored with Jennifer Quinn) *Proofs That Really Count: The Art of Combinatorial Proof*, published by the MAA. In 2006, that book received the MAA's Beckenbach Book Prize. From 2004 to 2008, Professors Benjamin and Quinn served as the coeditors of *Math Horizons* magazine, which is published by the MAA and enjoyed by more than 20,000 readers, mostly undergraduate math students and their teachers. In 2009, the MAA published Professor Benjamin's latest book, *Biscuits of Number Theory*, coedited with Ezra Brown.

Professor Benjamin is also a professional magician. He has given more than 1000 "mathemagics" shows to audiences all over the world (from primary schools to scientific conferences), in which he demonstrates and explains his calculating talents. His techniques are explained in his book *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*. Prolific math and science writer Martin Gardner calls it "the clearest, simplest, most entertaining, and best book yet on the art of calculating in your head." An avid game player, Professor Benjamin was winner of the American Backgammon Tour in 1997.

Professor Benjamin has appeared on dozens of television and radio programs, including the *Today* show, *The Colbert Report*, CNN, and National Public Radio. He has been featured in *Scientific American*, *Omni*, *Discover*, *People*, *Esquire*, *The New York Times*, the *Los Angeles Times*, and *Reader's Digest*. In 2005, *Reader's Digest* called him "America's Best Math Whiz." ■

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INTRODUCTION

LECTURE GUIDES

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The Secrets of Mental Math

Scope:

Most of the mathematics that we learn in school is taught to us on paper with the expectation that we will solve problems on paper. But there is joy and lifelong value in being able to do mathematics in your head. In school, learning how to do math in your head quickly and accurately can be empowering. In this course, you will learn to solve many problems using multiple strategies that reinforce number sense, which can be helpful in all mathematics courses. Success at doing mental calculation and estimation can also lead to improvement on several standardized tests.

We encounter numbers on a daily basis outside of school, including many situations in which it is just not practical to pull out a calculator, from buying groceries to reading the newspaper to negotiating a car payment. And as we get older, research has shown that it is important to find activities that keep our minds active and sharp. Not only does mental math sharpen the mind, but it can also be a lot of fun.

Our first four lectures will focus on the nuts and bolts of mental math: addition, subtraction, multiplication, and division. Often, we will see that there is more than one way to solve a problem, and we will motivate many of the problems with real-world applications.

Once we have mastery of the basics of mental math, we will branch out in interesting directions. Lecture 5 offers techniques for easily finding approximate answers when we don't need complete accuracy. Lecture 6 is devoted to pencil-and-paper mathematics but done in ways that are seldom taught in school; we'll see that we can simply write down the answer to a multiplication, division, or square root problem without any intermediate results. This lecture also shows some interesting ways to verify an answer's correctness. In Lecture 7, we go beyond the basics to explore advanced multiplication techniques that allow many large multiplication problems to be dramatically simplified.

In Lecture 8, we explore long division, short division, and Vedic division, a fascinating technique that can be used to generate answers faster than any method you may have seen before. Lecture 9 will teach you how to improve your memory for numbers using a phonetic code. Applying this code allows us to perform even larger mental calculations, but it can also be used for memorizing dates, phone numbers, and your favorite mathematical constants. Speaking of dates, one of my favorite feats of mental calculation is being able to determine the day of the week of any date in history. This is actually a very useful skill to possess. It’s not every day that someone asks you for the square root of a number, but you probably encounter dates every day of your life, and it is quite convenient to be able to figure out days of the week. You will learn how to do this in Lecture 10.

In Lecture 11, we venture into the world of advanced multiplication; here, we’ll see how to square 3- and 4-digit numbers, find approximate cubes of 2-digit numbers, and multiply 2- and 3-digit numbers together. In our final lecture, you will learn how to do enormous calculations, such as multiplying two 5-digit numbers, and discuss the techniques used by other world-record lightning calculators. Even if you do not aspire to be a grandmaster mathemagician, you will still benefit tremendously by acquiring the skills taught in this course. ■

The Art of Guesstimation

Lecture 5

Your body is like a walking yardstick, and it's worth knowing things like the width of your hand from pinkie to thumb, or the size of your footsteps, or parts of your hand that measure to almost exactly one or two inches or one or two centimeters.

Mental estimation techniques give us quick answers to everyday questions when we don't need to know the answer to the last penny or decimal point. We estimate the answers to addition and subtraction problems by rounding, which can be useful when estimating the grocery bill. As each item is rung up, round it up or down to the nearest 50 cents.

To estimate answers to multiplication or division problems, it's important to first determine the order of magnitude of the answer. The general rules are as follows:

- For a multiplication problem, if the first number has x digits and the second number has y digits, then their product will have $x + y$ digits or, perhaps, $x + y - 1$ digits. Example: A 5-digit number times a 3-digit number creates a 7- or 8-digit number.

- To find out if the answer to $a \times b$ will have the larger or smaller number of digits, multiply the first digit of each number. If that product is 10 or more, then the answer will be the larger number. If that product is between 5 and 9, then the answer could go either way. If the product is 4 or less, then the answer will be the smaller number.

- For a division problem, the length of the answer is the difference of the lengths of the numbers being divided or 1 more. (Example: With an 8-digit number divided by a 3-digit number, the answer will have $8 - 3 = 5$ or 6 digits before the decimal point.)

- To find out how many digits come before the decimal point in the answer to $a \div b$, if the first digit of a is the same as the first digit of b, then compare the second digits of each number. If the first digit of a is larger than the first digit of b, then the answer will be the longer choice. If the first digit of a is less than the first digit of b, then the answer will be the shorter choice.

In estimating sales tax, if the tax is a whole number, such as 4%, then estimating it is just a straight multiplication problem. For instance, if you're purchasing a car for $23,456, then to estimate 4% tax, simply multiply 23,000 × 0.04 (= $920; exact answer: $938). If the tax is not a whole number, such as 4.5%, you can calculate it using 4%, but then divide that amount by 8 to get the additional 0.5%.

To estimate answers to multiplication or division problems, it's important to first determine the order of magnitude of the answer.

Suppose a bank offers an interest rate of 3% per year on its savings accounts. You can find out how long it will take to double your money using the "Rule of 70"; this calculation is 70 divided by the interest rate.

Suppose you borrow $200,000 to buy a house, and the bank charges an interest rate of 6% per year, compounded monthly. What that means is that the bank is charging you 6/12%, or 1/2%, interest for every month of your loan. If you have 30 years to repay your loan, how much will you need to pay each month? To estimate the answer, follow these steps:

- Find the total number of payments to be made: 30 × 12 = 360.

- Determine the monthly payment without interest: $200,000 ÷ 360. Simplify the problem by dividing everything by 10 (= 20,000 ÷ 36), then by dividing everything by 4 (= 5000 ÷ 9, or 1000 × 5/9). The fraction 5/9 is about 0.555, which means the monthly payment *without interest* would be about 1000 × 0.555, or $555.

- Determine the amount of interest owed in the first month: \$200,000 × 0.5% = \$1000.

A quick estimate of your monthly payment, then, would be \$1000 to cover the interest plus \$555 to go toward the principal, or \$1555. This estimate will always be on the high side, because after each payment, you'll owe the bank slightly less than the original amount.

Square roots arise in many physical and statistical calculations, and we can estimate square roots using the divide-and-average method. To find the square root of a number, such as 40, start by taking any reasonable guess. We'll choose $6^2 = 36$. Next, divide 40 by 6, which is 6 with a remainder of 4, or 6 2/3. In other words, 6 × 6 2/3 = 40. The square root must lie between 6 and 6 2/3. If we average 6 and 6 2/3, we get 6 1/3, or about 6.33; the exact answer begins 6.32! ■

Important Term

square root: A number that, when multiplied by itself, produces a given number. For example, the square root of 9 is 3 and the square root of 2 begins 1.414…. Incidentally, the square root is defined to be greater than or equal to zero, so the square root of 9 is *not* –3, even though –3 multiplied by itself is also 9.

Suggested Reading

Benjamin and Shermer, *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*, chapter 6.

Doerfler, *Dead Reckoning: Calculating Without Instruments.*

Hope, Reys, and Reys, *Mental Math in the Middle Grades.*

Kelly, *Short-Cut Math.*

Ryan, *Everyday Math for Everyday Life: A Handbook for When It Just Doesn't Add Up.*

Weinstein and Adam, *Guesstimation: Solving the World's Problems on the Back of a Cocktail Napkin*.

Problems

Estimate the following addition and subtraction problems by rounding each number to the nearest thousand, then to the nearest hundred.

1. 3764 + 4668

2. 9661 + 7075

3. 9613 – 1252

4. 5253 – 3741

Estimate the grocery total by rounding each number up or down to the nearest half dollar.

5.	**6.**	**7.**
5.24	0.87	0.78
0.42	2.65	1.86
2.79	0.20	0.68
3.15	1.51	2.73
0.28	0.95	4.29
0.92	2.59	3.47
4.39	1.60	2.65

What are the possible numbers of digits in the answers to the following problems?

8. 5 digits times 3 digits

9. 5 digits divided by 3 digits

10. 8 digits times 4 digits

11. 8 digits divided by 4 digits

For the following problems, determine the possible number of digits in the answers. (Some answers may allow two possibilities.) A number written as 3abc represents a 4-digit number with a leading digit of 3.

12. 3abc × 7def

13. 8abc × 1def

14. 2abc × 2def

15. 9abc ÷ 5de

16. 1abcdef ÷ 3ghij

17. 27abcdefg ÷ 26hijk

18. If a year has about 32 million seconds, then 1 trillion seconds is about how many years?

19. The government wants to buy a new weapons system costing $11 billion. The U.S. has about 100,000 public schools. If each school decides to hold a bake sale to raise money for the new weapons system, then about how much money does each school need to raise?

20. If an article is sent to two independent reviewers, and one reviewer finds 40 typos, the other finds 5 typos, and there were 2 typos in common, then estimate the total number of typos in the document.

21. Estimate 6% sales tax on a new car costing $31,500. Adjust your answer for 6.25% sales tax.

22. To calculate 8.5% tax, you can take 8% tax, then add the tax you just computed divided by what number? For 8.75% tax, you can take 9% tax, then subtract that tax divided by what number?

23. If money earns interest compounded at a rate of 2% per year, then about how many years would it take for that money to double?

24. Suppose you borrow $20,000 to buy a new car, the bank charges an annual interest rate of 3%, and you have 5 years to pay off the loan. Determine an underestimate and overestimate for your monthly payment, then determine the exact monthly payment.

25. Repeat the previous problem, but this time, the bank charges 6% annual interest and gives you 10 years to pay off the loan.

26. Use the divide-and-average method to estimate the square root of 27.

27. Use the divide-and-average method to estimate the square root of 153.

28. Speaking of 153, that's the first 3-digit number equal to the sum of the cubes of its digits ($153 = 1^3 + 5^3 + 3^3$). The next number with that property is 370. Can you find the third number with that property?

Solutions for this lecture begin on page 108.